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Diffusion Transforms

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1. INTRODUCTION

In [16] we found some results about Itô's stochastic integral by using the transform $H \to \hat{H}$, $\hat{H}(t, x) = EH(y_t, x)$, where y_t is the stochastic integral and H is a solution to a hyperbolic partial differential equation. It turns out that $\hat{H}$ then satisfies a parabolic partial differential equation or inequality. In the present paper we carry this idea further.

In Section 2 we show that, when y_t is replaced by a general diffusion process, the above transform can be used to map solutions of a second-order abstract Cauchy problem to solutions of first-order abstract Cauchy problems. In Section 3 the same idea is used to give a new probabilistic derivation of the formulas for the semigroup generated quadratic polynomials $P(A)$ in terms of Gaussian integrals of the semigroup generated by A. In Section 4, Kac's solution of the telegraph equation (see, e.g. [13]) is worked out explicitly in the singular case of the Darboux equation, and this representation is combined with results of Section 1 to yield a probabilistic derivation of a formula for solutions to the Darboux equation.

It may be helpful to point out the difference between our use of a diffusion, as a *time*, and the more standard use, as a *space* variable. Specifically, if φ is the initial value of the solution of a parabolic equation governing a diffusion x_t, $E\varphi(x_t)$ is a solution to the equation. But we instead consider $Eu(x_t)$ where $t \to u(t)$ is already a solution to a certain type of abstract Cauchy problem, and x_t is a diffusion related in a certain way to that abstract Cauchy problem. The function $t \to Eu(x_t)$ then satisfies a new abstract Cauchy problem.

2. TRANSFORMING SECOND-ORDER ACP TO FIRST-ORDER ACP

Suppose u is a solution to the abstract Cauchy problem (ACP)

$$\tfrac{1}{2}e^2(t)u_{tt} + f(t)u_t = Au \qquad t \in R$$

where $u(t)$ lies in a Banach space B and A is a linear operator on B. (For the moment, further assumptions on A are unnecessary, but we might as well assume A to be a C_0-semi-group generator.) Let $(x_t, t \geqslant 0)$ be the diffusion process satisfying the stochastic differential equation [14; p. 50]

$$\begin{cases} dx_t = f(x_t)\,dt + e(x_t)\,db_t \\ x_0 = 0 \end{cases}$$

where db_t is the differential of a standard Brownian motion process. (We assume that e and f are C' and satisfy a growth condition sufficient to insure that the above equation has a solution for all t, e.g. $e^2 + f^2 \leqslant K(1 + t^2)$.) Next, define the "diffusion" transform

$$\hat{u}(t) = Eu(x_t) \qquad t \geqslant 0.$$

(Note that this is a Bochner integral.) By Itô's lemma [14; p. 32] (for smooth B-valued functions u),

$$\begin{aligned} du(x_t) &= u_t(x_t)\,dx_t + \tfrac{1}{2}u_{tt}(x_t)(dx_t)^2 \\ &= [u_t(x_t)f(x_t) + \tfrac{1}{2}u_{tt}(x_t)\,e^2(x_t)]\,dt + u_t(x_t)\,e(x_t)\,db_t \\ &= Au(x_t)\,dt + u_t(x_t)\,e(x_t)\,db_t\,. \end{aligned}$$

Thus under the added condition

$$E\int_0^s u_t^2(x_t)e^2\,(x_t)\,dt < \infty \qquad s \geqslant 0$$

we have

$$\begin{aligned} Eu(x_t) - u(0) &= E\int_0^t Au(x_s)\,ds \\ &= \int_0^t AEu(x_s)\,ds \end{aligned}$$

(E and A commute, as they act on different variables) and hence

$$\begin{cases} \hat{u}_t = A\hat{u} \\ \hat{u}(0) = u(0) \end{cases} \tag{2}$$

which proves that under the condition (1), *the transform* $\hat{u}$ *satisfies the first-order* ACP (2) *and moreover has the same initial value as* u.

We can also from (2) obtain an estimate on the way in which $\sup_t \| u(t)\|$ changes under a singular perturbation.

THEOREM. *Suppose that the abstract Cauchy problem*

$$\begin{aligned} \tfrac{1}{2}e^2(t)u_{tt} + u_t &= Au \qquad t \in R \\ u(0) &= \varphi \\ u_t(0) &= \psi \end{aligned}$$

possess a unique solution satisfying (1) *where* e *is* C' *and* $e^2(t) \leqslant K(1 + t^2)$ *for some* $K > 0$. *Then, no matter how small* e *is,*

$$\sup_{t \in R} \| u(t) \|$$

is always greater than or equal to the corresponding quantity for $e = 0$, *namely*

$$\sup_{t > 0} \| \hat{u}(t) \|$$

where $\hat{u}$ *is the unique solution to*

$$\begin{aligned} \hat{u}_t &= A\hat{u} \\ \hat{u}(0) &= \varphi. \end{aligned}$$

Proof. Follows trivially from $\hat{u}(t) = Eu(x_t)$ and the fact that $| Eu(x_t)| \leqslant E \,| u(x_t)|$.

We now give some applications of the diffusion transform.

EXAMPLE 1. If $f = 0$ and $e = 2^{1/2}$ then $x_t = 2^{1/2}b_t \overset{\text{law}}{=} b(2t)$, so

$$\hat{u}(t) = \int_R u(x) \frac{e^{-(x^2/4t)}}{(4\pi t)^{\frac{1}{2}}} \, dx.$$

The fact that the function $\hat{u}$ defined by this formula is a solution to $\hat{u}_t = A\hat{u}$, when $u_{tt} = Au$, has been noted numerous times [3, 11, 18] and was also mentioned in [16; Remark 2]. See also [1]. (In this manner the Poisson integral solution to the heat equation is derived from the solution to the wave equation having the same initial value.)

EXAMPLE 2. If $e = 2^{1/2}$ and $f(t) = 2\lambda t^{-1}$, $\lambda > 0$, then $(x_t \,, t \geqslant 0)$ is the (positive) Bessel process in $1 + 2\lambda$ "dimensions" (with a certain scaling of time)

$$\begin{aligned} dx_t &= 2\lambda x_t^{-1} \, dt + 2^{\frac{1}{2}} \, db_t \qquad & t > 0 \\ x_0 &= 0 & x_t \geqslant 0 \end{aligned} \tag{3}$$

and since (see [12; p. 60])

$$\begin{aligned} P(x_t \in dz) &= c(\lambda) t^{-(1-2\lambda)/2} \, e^{-z^2/4t} \, z^{2\lambda} \, dz \qquad z > 0 \\ c(\lambda)^{-1} &= 2^{2\lambda} \, \Gamma(\lambda + \tfrac{1}{2}) \end{aligned} \tag{4}$$

($x_t > 0$ almost surely for $t > 0$) we have that if u is a solution to the Darboux equation

$$u_{tt} + \frac{2\lambda}{t} u_t = Au \qquad t > 0 \tag{5}$$

then

$$\hat{u}_t = A\hat{u}$$

where

$$\hat{u}(t) = c(\lambda) \int_0^\infty u(x) t^{-(1+2\lambda)/2} e^{-x^2/4t} x^{2\lambda} \, dx.$$

Example 3. Finally we give a singular case of the diffusion transform. Suppose that

$$\begin{aligned} u_{tt} &= Au \qquad t > 0 \\ u(0) &= \varphi \\ u_t(0) &= \psi. \end{aligned} \tag{6}$$

We claim that if

$$\hat{u}(t) := Eu(2^{1/2} \mid b_t \mid) \qquad t > 0 \tag{7}$$

then

$$\begin{aligned} \hat{u}_t &= A\hat{u} + (\pi t)^{-1/2}\psi \qquad t > 0 \\ \hat{u}_0 &= \varphi. \end{aligned} \tag{8}$$

Our argument will be a formal one, but since (8) can be easily verified by a direct calculation from (7), we do not bother to make it rigorous. First, if $x_t = 2^{1/2} \mid b_t \mid$, then

$$dx_t = 2^{1/2} \operatorname{sgn}(b_t) \, db_t + 2^{1/2}\delta(b_t) \, dt$$

where δ = Dirac delta situated at zero. This follows by formally applying Itô's lemma to the continuous function $r \to 2^{1/2} \mid r \mid$ and interpreting derivatives distributionally.

Then a second application of Itô's lemma yields, formally,

$$\begin{aligned} du(x_t) &= u_t \, dx_t + \tfrac{1}{2} u_{tt} (dx_t)^2 \\ &= u_t(2^{1/2} \mid b_t \mid) \, 2^{1/2}\delta(b_t) \, dt + u_{tt}(2^{1/2} \mid b_t \mid) \, dt \\ &\quad + \text{a term of zero mean} \\ &= 2^{1/2} u_t(0) \, \delta(b_t) \, dt + Au(x_t) \, dt \\ &\quad + \text{a term of zero mean,} \end{aligned} \tag{9}$$

since $g(x)\,\delta(x) = g(0)\,\delta(x)$. It is simple to see that

$$E\delta(b_t) = (2\pi t)^{-1/2}$$

so (9) implies (8), upon taking expectations.

Note that if $\psi = 0$ then $\hat{u}$ satisfies $\hat{u}_t = A\hat{u}$. This, together with the fact that in (6) we assume only that u satisfies $u_{tt} = Au$ *for positive times*, suggests an application to non-well-posed problems. For example, take $A = -d^2/dx^2$, $\psi = 0$. Then $\hat{u}$, given by (7), is a solution of the backward heat equation. See also [2].

Remark 1. It should be noted that a similar diffusion transform method works if there are several coordinates $t_1, \ldots, t_n$ considered as time coordinates. I.e., if

$$\tfrac{1}{2} \sum_{i,j,k} e_{ij}(t)\, e_{kj}(t) \frac{\partial^2 u}{\partial t_i \partial t_j} + \sum_j f_j(t) \frac{\partial u}{\partial t_j} = Au \qquad \{t = (t_1, \ldots, t_n)\} \tag{10}$$

and

$$\begin{aligned} dx(r) &= f(x(r))\, dr + e(x(r))\, db_r \qquad r > 0 \\ x(0) &= 0 \in R^n \end{aligned}$$

then the diffusion transform $\hat{u}(r) = Eu(x(r))$ satisfies $\hat{u}_r = A\hat{u}$. (Here b_r, $r > 0$, is an n-dimensional Brownian motion and f is a mapping from R^n to R^n, and $e(x)$ is for each x in R^n the $n \times n$ matrix (e_{ij}) in equation (10).) Of course a condition such as (1) must be satisfied.

Remark 2. Our fundamental result (2) can also be proved by more classical means. The following is a formal classical proof. Let $p(t, x, s)$ be the positive solution to

$$p_t = H_s{}^* p = H_x p$$

and $p(0, x, s) = \delta(x - s)$,

$$\int_{-\infty}^{\infty} p(t, x, s)\, ds = 1,$$

where

$$H_s = \frac{1}{2} e^2(s) \frac{d^2}{ds^2} + f(s) \frac{d}{ds},$$

and $H_s{}^*$ is the formal adjoint.

Then if we define

$$\hat{u}(t) = \int_{-\infty}^{\infty} u(s) p(t, 0, s)\, ds,$$

formally we have

$$\hat{u}_t(t) = \int_{-\infty}^{\infty} up_t\, ds = \int_{-\infty}^{\infty} uH_s{}^*p\, ds$$
$$= \int_{-\infty}^{\infty} (H_s u)p\, ds$$
$$= \int_{-\infty}^{\infty} (Au)p\, ds$$
$$= A\int_{-\infty}^{\infty} up\, ds = A\hat{u}(t),$$

and since $p(0, x, s) = \delta(x - s)$,

$$\hat{u}(0) = \int_{-\infty}^{\infty} u(s)\delta(x) = u(0).$$

3. Quadratic Polynomials of A

Suppose that A is the generator of a strongly continuous group $(T_t, t \in R)$ of operators on the Banach space B, and suppose that u is the solution to the Cauchy problem

$$u_t = Au$$
$$u(0) = \varphi$$

i.e., $u(t) = T_t\varphi$. Now let $\beta > 0$, $\alpha \in R$. We show how to construct, by a diffusion transform, the semigroup generated by $\alpha A + \beta A^2$. (See [9–11, 15, 17, 19].) By Itô's Lemma,

$$\begin{aligned} du(\alpha t + (2\beta)^{\frac{1}{2}} b_t) &= (\alpha u_t + \beta u_{tt})\, dt + \text{a term of zero mean} \\ &= (\alpha A + \beta A^2)u\, dt + \text{a term of zero mean,} \end{aligned} \tag{11}$$

hence if $\hat{u}(t) := Eu(\alpha t + (2\beta)^{1/2}\, b_t)$ $(t > 0)$ we have

$$\begin{aligned} \hat{u}_t &= (\alpha A + \beta A^2)\hat{u} \\ \hat{u}(0) &= \varphi \end{aligned} \tag{12}$$

i.e., for $t > 0$

$$\begin{aligned} T_{P(A)}(t) &= ET_A(\alpha t + (2\beta)^{1/2} b_t) \\ &= \int_R T_A(\alpha t + (2\beta)^{1/2} x)\, \frac{e^{-(x^2/2t)}}{(2\pi t)^{\frac{1}{2}}}\, dx, \end{aligned} \tag{13}$$

where $P(A) = \alpha A + \beta A^2$.

Of course for the above to make sense φ (in (12)) must be in the domain of A^2, the condition guaranteeing zero mean in (11) must be satisfied, and strictly speaking an element $\psi \in \mathscr{D}(A^2)$ must be inserted on both sides of (13) to make a strong integral.

4. The Darboux Equation

If u is a solution to the Cauchy problem

$$\begin{aligned} u_{tt} &= Au \\ u(0) &= \varphi \\ u_t(0) &= \psi \end{aligned} \tag{14}$$

then under mild conditions on A the transform

$$\tilde{u}(t) = Eu(T(t)) \tag{15}$$

satisfies the Cauchy problem

$$\begin{aligned} \tilde{u}_{tt} + 2a(t)\tilde{u}_t &= A\tilde{u} \\ \tilde{u}(0) &= \varphi \\ \tilde{u}_t(0) &= \psi \end{aligned} \tag{16}$$

if $T(t)$ is the random time

$$T(t) := \int_0^t (-1)^{Q(r)}\, dr \tag{17}$$

where $(Q(r), r \geqslant 0)$ is a Poisson process with intensity $a(t)$, a being positive and smooth. This means

$$P(Q_r - Q_s = k) = e^{-\int_s^r a(x)\,dx} \left\{ \int_s^r a(x)\, dx \right\}^k \Big/ k! \tag{18}$$

for $r > s$ and $k = 0, 1, 2, \ldots$. This transformation was used by Kac (in a special case) and was later extended by Kaplan [13]. See also [9] for an explanation of the mechanism that brings this about.

The diffusion transform technique of Section 2 allows us to establish a connection between the integral (17) and the "time"-diffusion associated with (16). Namely:

Proposition. *Let a be a positive smooth function and let y_t be the diffusion*

$$\begin{aligned} dy_t &= 2a(y_t)\, dt + 2^{1/2}\, db_t \qquad t > 0 \\ y_0 &= 0. \end{aligned}$$

Let $(Q(r), r \geqslant 0)$ be a Poisson process with intensity a as in (18) *and assume Q constructed so as to be independent of $(b_t, t \geqslant 0)$. Then for $t > 0$*

$$T(y_t) = \int_0^{y_t} (-1)^{Q(r)}\, dr \overset{\text{law}}{=} b(2t).$$

Proof. We have seen in Section 2 that $t \to E\tilde{u}(y_t)$ and $t \to Eu(2^{1/2}b_t)$ both satisfy the Cauchy problem

$$v_t = Av \qquad t > 0$$
$$v(0) = \varphi$$

hence if A generates a strongly continuous semigroup, which we now assume, we must have (by uniqueness)

$$E\tilde{u}(y_t) = Eu(2^{1/2}b_t) \qquad t > 0,$$

but

$$E\tilde{u}(y_t) = Eu(T(y_t))$$

since Q and y are independent, so

$$Eu(T(y_t)) = Eu(2^{1/2}b_t)$$

and since it is true for *all* solutions u, it follows that

$$T(y_t) \overset{\text{law}}{=} 2^{1/2}\, b_t \overset{\text{law}}{=} b(2t),$$

which completes the proof.

We now discuss the transform of Kac–Kaplan in the singular case of the Darboux equation. We aim to find an explicit formula for $\tilde{u}$ in (15) so that $\tilde{u}$ satisfies the Darboux equation, i.e., equation (16) with $a(t) = \lambda t^{-1}$ where $\lambda > 0$. (Equation (16) then becomes equation (5).) The argument that follows will be formal at points. There is no reason to fill in any of the details since the final formula can be easily checked directly.

Let n be a positive integer and consider the process Q with intensity $n\lambda(nt + 1)^{-1}$. Note this converges to λt^{-1} as $n \to \infty$. (If we were to take $a(t) = \lambda t^{-1}$ directly then (18) would imply $Q(r) = \infty$ almost surely for all $r > 0$.) Let $T_n(t)$ be the resulting random time. We claim that

$$T_n(t) \overset{\text{law}}{\to} tZ \qquad t > 0 \tag{19}$$

as $n \to \infty$, where Z is a random variable satisfying $-1 \leqslant Z \leqslant 1$ and whose distribution we shall determine by using the Proposition. First note from (18) that

$$P(Q_n(r) = k) = (1 + nr)^{-\lambda} \frac{\lambda^k}{k!} (\log(1 + nr))^k. \tag{20}$$

Then

$$\begin{aligned} \frac{T_n(t)}{t} &= \frac{1}{t} \int_0^t (-1)^{Q_n(r)}\, dr \\ &= \frac{1}{nt} \int_0^{nt} (-1)^{Q_n(s/n)}\, ds \\ &\overset{\text{law}}{=} \frac{1}{nt} \int_0^{nt} (-1)^{Q_1(s)}\, ds \end{aligned} \tag{21}$$

since (20) shows $Q_n(s/n) \overset{\text{law}}{=} Q_1(s)$. Thus the problem arises to show that (21) possesses a limiting distribution as $nt \to \infty$. This can be shown by considering the moments of the right side of (21). Their limiting values can be easily computed by writing the powers as iterated integrals and observing, e.g., for the third moment, that

$$(-1)^{Q(r)+Q(s)+Q(t)} = (-1)^{Q(r)}(-1)^{Q(t)-Q(s)}$$

so that if $r < s < t$, the two factors on the right are independent. (This trick was used in [13].) Thus we regard (19) as established, and proceed with the determination of the law of Z. By the Proposition, if y_t is the Bessel process (3), independent of the Q-process, hence independent of Z,

$$y_t Z \overset{\text{law}}{=} b(2t) \qquad t > 0. \tag{22}$$

Let g be the density function of Z (g is clearly an even function) and let P_t be the density function of y_t (see (7)). Then (22) states

$$\int_0^\infty P_t\left(\frac{m}{r}\right) g(r) \frac{dr}{r} = \frac{e^{-(m^2/4t)}}{(4\pi t)^{\frac{1}{2}}} \qquad m > 0.$$

This is an integral equation for g which can be solved by taking the Mellin transform of both sides. The result is

$$P_t^*(s)\, g^*(s) = 2^{s-1} t^{s/2} \Gamma(s/2)$$

where the star indicates the Mellin transform. This yields, when P_t^* is computed

$$g^*(s) = \frac{1}{(4\pi)^{\frac{1}{2}}} \frac{\Gamma(s/2)\,\Gamma(\lambda + 1/2)}{\Gamma(\lambda + s/2)}$$

and from [8, p. 311] we find

$$g(r) = K(\lambda)(1 - r^2)^{\lambda-1} \qquad -1 \leqslant r \leqslant 1$$

$K(\lambda)$ being determined by $\int g = 1$.

The conclusion of all this is that if u satisfies (14) *then*

$$\tilde{u}(t) = Eu(tZ) = \int_{-1}^{1} K(\lambda)(1 - r^2)^{\lambda-1}\, u(tr)\, dr \tag{23}$$

satisfies

$$\tilde{u}_{tt} + \frac{2\lambda}{t}\tilde{u}_t = A\tilde{u}$$

with the same Cauchy data as u. This formula (23) may be found in numerous places, e.g. [4]; but as far as we know, this is the first probabilistic derivation.

5. Acknowledgment

Many interesting applications of Darboux-like Cauchy problems are studied in the papers of J. Donaldson [5–7] and it was through a discussion with J. Donaldson that I became interested in these problems.

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